LINEAR PROGRAMMING

After completing this worksheet and the related one, Ma5/10b(ii), you should be able to interpret diagrams such as those used in Critical Path Analysis or Linear Programming. (Ma5/10b)

This unit shows how the important technique of Linear Programming is used to solve practical problems in which you want to **maximise** or **minimise** a quantity, but subject to satisfying a number of constraints. The technique was developed during the Second World War, but has since found applications in most branches of management and planning.

Before looking at the technique, the topic of graphing inequalities (Ma3/8b) will be revised.

Example 1

Find the region satisfied by all the inequalities

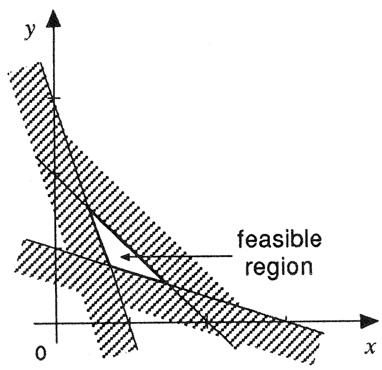
$$\begin{aligned} x+y &\le 2 \\ 3x+y &\ge 3 \\ x+3y &\ge 3. \end{aligned}$$

Solution

You first draw the corresponding equalities, namely

$$\begin{aligned} x+y &= 2 \\ 3x+y &= 3 \\ x+3y &= 3. \end{aligned}$$

You shade the area which is **not** allowed.



The diagram shows the region, called the **feasible region**, in which all the inequalities are satisfied.

What region is defined if $x+y \le 2$ is replaced by $x+y \ge 2$?

Activity 1

(a) Find the region which satisfies all the inequalities

$$\begin{aligned} x+y &\ge 2 \\ x+4y &\le 4 \\ y &\ge -1 \end{aligned}$$

(b) Write down three different linear equations of the form $ax+by=c$.

Which three inequalities are satisfied in the finite region formed by these lines?

The technique of Linear Programming is illustrated in the next example.

Example 2

A farmer has 20 hectares for growing barley and swedes. He has to decide how much of each to grow. The cost per hectare for barley is £30 and for swedes is £20. The farmer has budgeted £480.

Barley requires 1 man-day per hectare and swedes require 2 man-days per hectare. There are 36 man-days available.

The profit on barley is £100 per hectare and on swedes is £120 per hectare.

Find the number of hectares of each crop the farmer should sow to maximise profits.

Solution

First define

(a) **Unknowns** to be found:

$x =$ number of hectares of barley

$y =$ number of hectares of swedes

(b) **Constraints** to be satisfied:

Land : $x + y \le 20$

Cost : $30x + 20y \le 480$

Manpower : $x + 2y \le 36$

(c) **Profit function** to maximise:

$C = 100x + 120y$

To summarise,

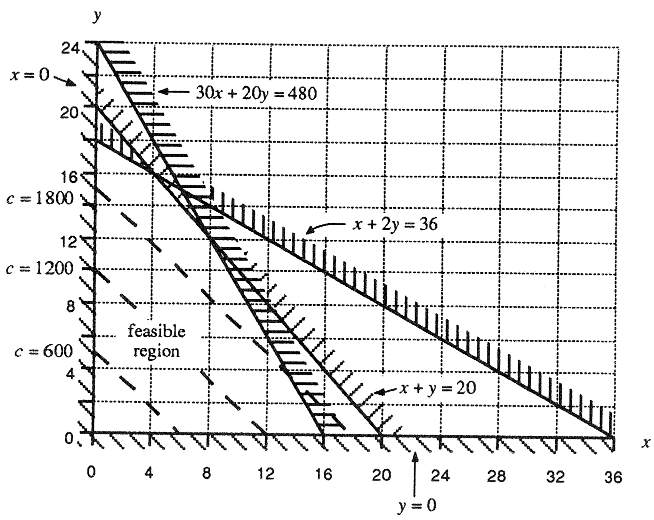
maximise $C = 100x + 120y$,

subject to

$$x + y \le 20$$
$$30x + 20y \le 480$$
$$x + 2y \le 36$$
$$x \ge 0$$
$$y \ge 0$$

(since you cannot have negative areas)

The **feasible region** is defined as the region enclosed by the five inequalities.



Profit lines, $C =$ constant, are also shown on the diagram; that is, the family of lines

$$C = 100x + 120y$$

for varying values of C. The value of C increases as the lines move to the right.

Now, continuing in this way, the maximum value of C will occur at the intersection of

$$x + 2y = 36 \text{ and } x + y = 20.$$

This gives $x = 4$, $y = 16$ and $C = 2320$.

So to maximise profit, the farmer should sow 4 hectares with barley and 16 hectares with swedes.

Here is another example to illustrate the technique.

Example 3

A firm manufactures two types of box, each requiring the same amount of material. They both go through a folding machine and a stapling machine.

Type A boxes require 4 seconds on the folding machine and 3 seconds on the stapling machine.

Type B boxes require 2 seconds on the folding machine and 7 seconds on the stapling machine.

Each machine is available for 1 hour. There is a profit of 40p on each **Type A** box and 30p on each **Type B** box.

How many of each type should be made to maximise the profit?

Solution

First define the **unknowns** :

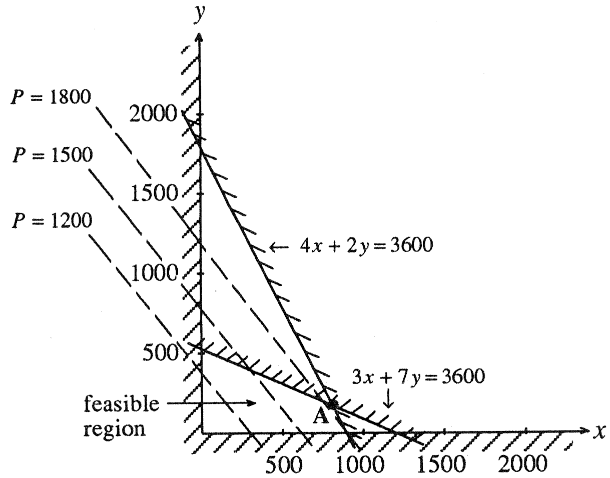
let $x =$ number of **Type A** boxes produced
$y =$ number of **Type B** boxes produced

Now the **constraints**:

for the folding machine $4x+2y \le 3600$

for the stapling machine $3x+7y \le 3600$

and, obviously, $x \ge 0,\ y \ge 0$.



These four constraints define the feasible region.

Finally, the **profit function** $P = 40x + 30y$.

This defines a family of straight lines for different values of P, shown dotted on the diagram.

The value of P on these lines increases as you move to the right. So the maximum value of P, with all the constraints being satisfied, will occur at the vertex A, defined by the intersection of

$$3x+7y = 3600$$

and $$4x+2y = 3600.$$

This gives $$x = \frac{9000}{11},\quad y = \frac{1800}{11},$$

and maximum value of P is $$P = 40 \times \frac{9000}{11} + 30 \times \frac{1800}{11} = \frac{414000}{11}.$$

Note that this solution does not give **integer** values for x and y, so that to find an exact solution, you would need to look carefully at integer solutions near the vertex but still in the feasible region. This is a much more difficult problem (known as integer programming) and it can be ignored here for simplicity.

By now, you should have noticed that, in general, the maximum (or minimum) of the profit (or cost) function occurs at a vertex. It could also occur along one of the boundary lines. So a simpler method is to evaluate the function at each vertex and from these values you can show the maximum (or minimum).

Activity 2

Repeat Example 2, but this time just evaluate the profit function at each vertex. Choose the vertex that gives the maximum value of C.

EXERCISES

1. Ann and Margaret run a small business in which they work together making blouses and skirts.

   Each blouse takes 1 hour of Ann's time together with 1 hour of Margaret's time. Each skirt involves Ann for 1 hour and Margaret for half an hour. Ann has 7 hours available each day and Margaret has 5 hours each day.

   They could just make blouses or they could just make skirts or they could make some of each.

   Their first thought was to make the same number of each. But they get £8 profit on a blouse and only £6 profit on a skirt.

   (a) Formulate the problem as a linear programming problem.

   (b) Find three solutions which satisfy the constraints.

2. A distribution firm has to transport 1200 packages using large vans which can take 200 packages each and small vans which can take 80 packages each.

   The cost of running each large van is £40 and of each small van is £20. Not more than £300 is to be spent on the job. The number of large vans must not exceed the number of small vans.

   Formulate this problem as a linear programming problem given that the objective is to **minimise** costs. Find the optimum solution.

3. A contractor hiring earth moving equipment has the choice of two machines. **Type A** costs £25 per day to hire, needs one man to operate it and moves 30 tonnes of earth per day. **Type B** costs £10 per day to hire, needs four men to operate it and moves 70 tonnes of earth per day. The contractor can spend up to £500 per day, has a labour force of 64 men available and can use a maximum of 25 machines on the site.

   Find the maximum weight of earth that the contractor can move in one day.

4. A firm manufactures wood screws and metal screws. All the screws have to pass through a threading machine and a slotting machine. A box of wood screws requires 3 minutes on the slotting machine and 2 minutes on the threading machine. A box of metal screws requires 2 minutes on the slotting machine and 8 minutes on the threading machine. In a week, each machine is available for 60 hours.

   There is a profit of £10 per box on wood screws and £17 per box on metal screws.

   Formulate this problem as a linear programming problem given that the objective is to **maximise** profit.

5. A factory employs unskilled workers earning £135 per week and skilled workers earning £270 per week. It is required to keep the weekly wage bill below £24300.

   The machines require a minimum of 110 operators, of whom at least 40 must be skilled. Union regulations require that the number of skilled workers should be at least half the number of unskilled workers.

   If x is the number of unskilled workers and y the number of skilled workers, write down all the constraints to be satisfied by x and y.

6. A landscape designer has £200 to spend on planting trees and shrubs to landscape an area of 1000 m^2. For a tree he plans to allow 25 m^2 and for a shrub 10 m^2. Planting a tree will cost £2 and a shrub £5.

   If he plants 30 shrubs what is the maximum number of trees he can plant?

   If he plants 3 shrubs for every tree, what is the maximum number of trees he can plant?